PRINCIPLES OF ACQUIRING INVARIANT IN MATHEMATICS TASK DESIGN: A DYNAMIC GEOMETRY EXAMPLE

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This paper is a theoretical discussion on a pedagogic task design model based on using variation as an epistemic tool. A set of four Principles of Acquiring Invariant is put forward that is complementary to the patterns of variation in Marton's Theory of Variation. These principles are then used as adhesive to tie together the epistemic modes in a model of task design in dynamic geometry environment that the author proposed in earlier research literature. A dynamic geometry task sequence is used to illustrate how the Principles of Acquiring Invariant can be used in mathematics task design.

INTRODUCTION

Marton's Theory of Variation is a theory of learning and awareness that asks the question: what are powerful ways to discern and to learn? In recent years, the theory has been applied in different pedagogical contexts (see for example, Lo, 2012; Lo & Marton, 2012). The Theory of Variation starts with a taken-for-granted observation: nothing is one thing only, and each thing has many features. In this theory, discernment is about how to go from a holistic experience of a phenomenon (e.g. seeing a forest) to separating out different features (e.g. seeing a tree) in the phenomenon (cf. Marton and Booth, 1997). It concerns with how to pick up meaningful experiences through our senses, and how meaning comes about from relationship between similarity and difference derived under simultaneous attention. In particular, there is a discernment ordering from difference to similarity. That is, learning and awareness begins with noticing difference before observing similarity. Suppose I can only perceive "grey" in certain situation, then "grey" has no meaning for me even if you show me a grey chair, a grey car, or a grey whatever. "Greyness" becomes meaningful to me only if I can perceive something else other than "grey". Thus, contrast (finding counter-examples focusing on difference) should come before generalization (which can be regarded as an inductive process focusing on similarity) in discernment. In this connection, a fundamental idea in the Theory of Variation is simultaneity. When we are simultaneously aware of (intentional focusing our attention on) different aspects of a phenomenon, we notice differences and similarities. By strategically observing variations of differences, similarities and their relationships, critical features of the phenomenon may be brought out. Morton proposed four patterns of variation as such strategic means: contrast, separation, generalization and fusion (Marton et al., 2004). A major undertaking of the Theory of Variation is to study how to organize and interpret a pedagogical event in powerful ways in terms of these patterns of variation (Lo & Marton, 2012).

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THE THEORY OF VARIATION IN MATHEMATICS CONCEPT DEVELOPMENT

In PME 27, I presented the first application of the Theory of Variation to mathematics pedagogy in dynamic geometry dragging exploration (Leung, 2003). There the four patterns of variation were used to interpret dragging modalities in a dynamic geometry construction problem to explore the gap between experimental reasoning and theoretical reasoning. This began a long programme of study where in my subsequent work; the values of the four patterns were gradually changing from originally as means to categorize possible powerful ways to discern into epistemic functions that can be used to bring about mathematical concept development (see for example, Leung, 2008; Leung, 2012; Leung et al., 2013). An epistemic activity in doing mathematics is to discern critical features (or patterns) in a mathematical situation. When these critical features are given interpretations, they may become invariants that can be used to conceptualize the mathematical situation. In Leung (2012), I used classification of plane figures as an example to develop a variation pedagogic model. The model consists of a sequence of discernment units in which different variation strategies are used to unveil different feature types of plane figure: intuitive visual type, geometrical property type, and equivalent geometrical properties type. Each discernment unit contains a process of mathematical concept development that is fused together by contrast and generalization driven by separation. The sequence represents a continuous process of refinement of mathematical concept, from primitive to progressively formal and mathematical. Mhlolo (2013) later used this model as an analytical framework to interpret a sequence of richly designed mathematics lessons teaching the conceptual development of number sequence. The upshot is, in variation perspective, mathematical concepts can be developed by strategic observation and variation interaction in terms of contrasting and comparing, separating out critical features, shifting focus of attention (cf. Mason, 1989) and varying features together to seek emergence of invariant patterns. A variation interaction is “a strategic use of variation to interact with a mathematics learning environment in order to bring about discernment of mathematical structure” (Leung, 2012). It is also a strategic way to observe a phenomenon focusing on variation and simultaneity. I interpret “interaction” in the sense that the acts of observing may involve direct or indirect manipulation of the mathematical object under studied.

PRINCIPLES OF ACQUIRING INVARIANT

Simultaneity is the epistemic crux of variation. The four patterns in Marton’s Theory of Variation are different types of simultaneous focus used to perceive differences and similarities which lead to unveiling of critical features of what is being observed. Looking for invariant in variation and using invariant to cope with variation are essences of mathematical concept development. A mathematical concept is in fact an invariant. For example, the basic concept of the number “three” is an invariant cognized out of myriad representation of “three-ness”. Thus in acquiring mathematical knowledge, to perceive and to understand invariant amidst variation are central

epistemic goals. Putting these together, I put forward a set of four *Principles of Acquiring Invariant* that are complementary to Marton's four patterns of variation in the context of mathematics concept development (the italic words are the four patterns of variation):

Difference and Similarity Principle (DS) *Contrasting* and comparing in order to perceive or *generalize* possible invariant features

Sieving Principle (SI) *Separating* under prescribed constraints or conditions in order to reveal ("make visual") critical invariant features or relationships

Shifting Principle (SHI) Focusing and paying attention to different or similar features of a phenomenon at different time or situations in order to discern *generalized* invariant

Co-variation Principle (CO) Co-varying or *fusing* together multiple features at the same time in order to perceive possible emergent pattern or invariant relationship between the features

These four principles work with the four patterns of variation in a concerted way. All four principles, just like the four patterns, are different aspects of simultaneity and contrast. They are cognitive activities to look for mathematical invariants leading to development of mathematical concept. In particular, they have the following predominant functions. DS is about contrast and generalization leading to awareness of perceptual invariant feature. SI is about awareness of hidden invariant feature that is being separated out under variation when only selected aspects of the phenomenon are allowed to vary. SHI is about diachronic (across time) simultaneity leading to possible generalization in the conjecture making process. CO is about synchronic (same time) simultaneity leading to fusing together of critical features in the mathematical concept formation process. These four principles are learner driven which can be cognitively mingled and nested together. During a variation interaction, a learner can apply these principles with different weight and transparency. In the next section, I will illustrate a pedagogical example of these principles in designing a sequence of dynamic geometry tasks.

MATHEMATICS TASK DESIGN: A DYNAMIC GEOMETRY EXAMPLE

In Leung (2011), I proposed an epistemic model of task design in dynamic geometry environment (DGE). It consists of a sequence of three nested epistemic mode of cognitive activities:

Practices Mode (PM) Construct DGE objects or manipulate pre-designed DGE objects. Interact with DGE feedbacks to develop (a) skill-based routines; (b) modalities of behaviour; (c) modes of situated dialogue.

Critical Discernment Mode (CDM) Observe, record, recognize and re-present (re-construct) DGE invariant.

Situated Discourse Mode (SDM) Develop reasoning that lead to making generalized DGE conjecture. Develop DGE discourses and modes of reasoning to explain and prove.

These task design modes are nested in the sense that CDM is a cognitive extension of PM and SDM is a cognitive extension of CDM. An exploration space is opening up for learners as the task sequence progresses to construct first-hand understanding of the mathematical concepts carried by the task. It is a nested expanding space where practices evolve into discernment followed by discernment evolves into discourses. Within each mode, cognitive activities can be organized by variation tasks. Thus, the four Principles of Acquiring Invariant can be used as a skeleton to frame this epistemic model of task design. A DGE task sequence can be designed combining the epistemic modes and the Principles of Acquiring Invariant to constitute an evolving process (not necessarily linear) that merges gradually from dominate perceptive experiential "thinking" to dominate conceptual theoretical "thinking". The following is an example of such a task sequence. It is conceptualized and designed by using a student DGE exploration studied in Leung, Baccaglini-Frank and Mariotti (2013).

TASK 1: Construction

PM: DGE Construction

Construct three points A, B, and C on the screen, the line through A and B, and the line through A and C. Construct a line l parallel to AC through B, and a line perpendicular to l through C. Label the point of intersection of these two lines D. Consider the quadrilateral ABCD (see Figure 1).

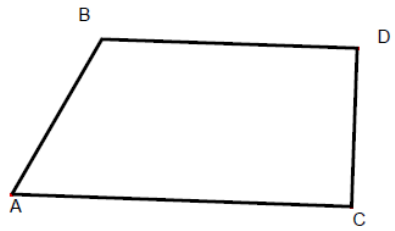

Figure 1

TASK 2: Contrast and Comparison

PM / DS: Variation tasks are used to bring about awareness of different and similar aspects/features in a DGE phenomenon that leads to observable invariants

2.1 Drag A, B, C to different positions to make different quadrilaterals

2.2 How many different or similar types of quadrilateral ABCD can you make?

2.3 Describe how you drag a point to make it changes into different types of quadrilateral

2.1 and 2.2 ask the learner to contrast and compare different positions of A, B and C as these vertices are being dragged to observe how many different types of quadrilateral can be formed. 2.3 ask the learner to think about the dragging strategies used to obtain different types of quadrilateral, thus motivating the learner to develop dragging skills and strategies, to relate feedback and dragging action, and to begin a DGE-based reasoning about perceiving DGE invariant. Figure 2 are two snapshots for different positions of A where B and C are fixed. There are only two types of possible quadrilaterals: right-angled trapezium and rectangle. This is making use of the Difference and Similarity Principle.

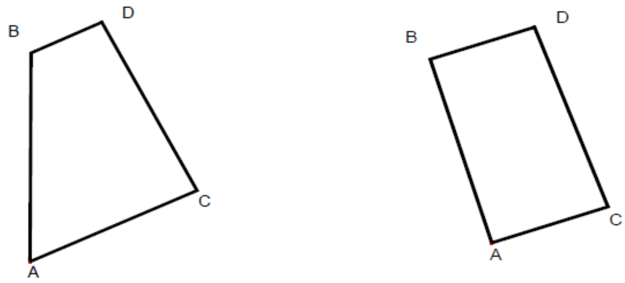

Figure 2

TASK 3: Separation of Critical Features

CDM / SI and SHI: Variation tasks are used to bring about awareness of critical (causal) relationship among the observed invariants

3.1 Activate the Trace function for point A. Drag A while keeping B and C fixed to maintain ABCD to look like a rectangle.

3.2 Describe your experience and what you observe

3.3 Make a guess on the geometrical shape of the path that A follows while maintaining ABCD to look like a rectangle. How do you make this guess? Call this guess a maintained-path (cf. Leung et al., 2013)

3.1 asks the learner to use a special function in DGE to record the trace of point A as it is being dragged to keep ABCD looks like a rectangle. Using rectangle as a perceptual invariant to constrain the dragging control makes visible the emergence of another perceptual invariant: the trace-mark of A which appears to take a geometrical shape (see Figure 3). Guessing and naming the trace motivates the learner to engage into a DGE discourse. This is making use of the Sieving Principle.

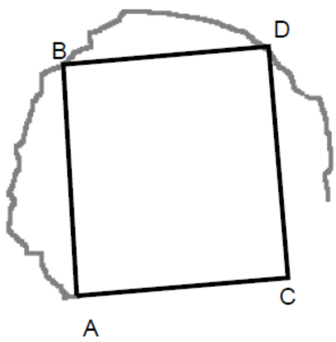

Figure 3

In 3.2 and 3.3, by asking the learner to describe his/her dragging experience and to make a guess on the geometrical shape of the traced path, the learner's cognitive mode is transiting from observation of DGE phenomena to discernment of critical features which could lead to concept formation. In particular, while the learner shifts his/her attention to the two perceptual invariants (the rectangular-like ABCD and the maintained-path) during dragging, attention to discern possible causal relationship between the two invariants may come about. This is the Shifting Principle.

TASK 4: Simultaneous Focus

SDM / SHI and CO: Variation tasks are used to bring about awareness of a connection between critical relationships observed and possible mathematical discourses (causal condition, formal/informal conjecture, concept, pattern, mathematical proof, etc.)

4.1 When A is being dragged to vary, vertices B, C and D either vary or not vary as consequence. Observe the behavior of B, C and D while A is varying to maintain ABCD looks like a rectangle.

4.2 Find a possible condition to relate the maintained-path and the varying configuration of B, C and D.

4.3 Use the condition found in 4.2 to construct the maintained path

4.1 and 4.2 are continuation of 3.3, the Shifting Principle continues with added attention to the consequential movements of the vertices A, B, C and D, thus the Co-variation Principle become in effect. In the process, the learner develops a DGE discourse for geometrical reasoning and construction. 4.3 is a consummation of the exploration in the form of a DGE soft construction (cf. Healy, 2000). The maintained-path takes the form of a circle centred at the midpoint of segment BC. The construction of this circle ensured D lies on the circle and when A is being dragged along this circle, ABCD becomes a rectangle (Figure 4).

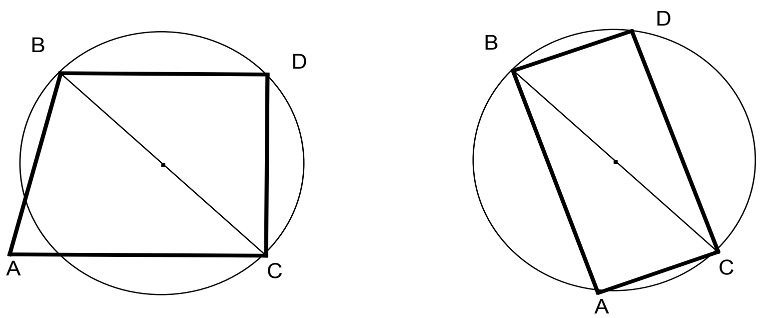

Figure 4

TASK 5: Conjecture and Proof (Development of Theoretical Reasoning)

SDM / CO: Development of DGE discourse to connect experimental reasoning and theoretical reasoning

5.1 Write a conjecture on what you have discovered in the form

GIVEN A DGE construction

IF (certain condition being maintained during dragging)

THEN (certain configuration appears to be maintained during dragging)

5.2 Drag A along the constructed maintained-path. Observe how different aspects of the figure vary together. Explain what you observe and formulate a logical argument to explain/prove your conjecture

4.3 (Figure 4) is a DGE representation of a conjecture, 5.1 asks the learner to write this in the form of a DGE-situated conditional statement, for example,

GIVEN Quadrilateral ABCD as constructed in TASK 1

IF A is being dragged along the circle centred at the midpoint of segment BC

THEN ABCD is always a rectangle

5.2 challenges the learner to formulate an explanation (or even a proof) for the conjecture just formed. I leave the readers to explore this discourse and to see how the Principles of Acquiring Invariant can be embedded in the reasoning process.

REMARKS

In the above I meshed together two epistemic frameworks, i.e. Principles of Acquiring Invariant and Task Design Epistemic Modes, to explore the mathematical concept formation process from experimental observation to discernment of abstraction using DGE as a context. A first remark is that these principles and epistemic modes form a nested network rather than follow a linear hierarchy. At any one instance during an exploration, any one of the principles and one of the modes can take dominance. These cognitive activities are pretty much learner driven but when designing a mathematical task, the designer can guide (as the five Tasks above) a learner to pay more attention to particular principle and mode while other principles and mode can be put in the cognitive background. This shifting between foreground and background is in fact a basic idea in the Theory of Variation. A second remark is that the task design model discussed in this paper is an attempt to crystalize a possible process bridging the experimental-theoretical gap in the DGE context. Specifically, the upshot of using variation and invariant is to drive an epistemic sequence that may look like:

> Constraint → Pattern Observation → Predictability → Emergence of Causal Relationship → Concept Formation → Explanation/Proof

This paper is an attempt to enrich the current research literature on the use of variation in mathematics education and to propose a perspective focusing on invariant that is pertinent to mathematics knowledge acquisition.

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